More about induction proofs
(a supplement for dProgSprog 2012)

Olivier Danvy <danvy@cs.au.dk>

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Contents

1 Miscellaneous notations

- The infix sign "≡" is read as "is defined to be", as in "$\mathbb{N} \equiv$ the set of natural numbers".
- Summing from 0 to a given natural number n:

  $$\sum_{i=0}^{n} i \equiv 0 + 1 + 2 + \ldots + (n-2) + (n-1) + n$$

  Equivalently, one can also write $\sum_{i=0}^{n} i$.

  In the lecture notes, $\sum_{i=0}^{n} i$ is written as "`sum for i = 0 to n of i`".
- Summing from a given natural number m to a given natural number n, where $m \leq n$:

  $$\sum_{i=m}^{n} i \equiv m + (m+1) + (m+2) + \ldots + (n-2) + (n-1) + n$$

  In the lecture notes, $\sum_{i=m}^{n} i$ is written as "`sum for i = m to n of i`".

- Summing from a given natural number m to the same natural number m (a border case):

$$\sum_{i=m}^{m} i \equiv m$$

In the lecture notes, $\sum_{i=m}^{m} i$ is written as "`sum for i = m to m of i`".

- Summing from a given natural number m to a given natural number n, where $m > n$ (a pathological case):

$$\sum_{i=m}^{n} i \equiv 0$$

- A nested sum, Version 1:

$$\sum_{i=m_1}^{n_1} \left(\sum_{j=m_2}^{n_2} (i+j) \right)$$

$$\equiv \sum_{j=m_2}^{n_2} (m_1+j) + \sum_{j=m_2}^{n_2} ((m_1+1)+j) + \ldots + \sum_{j=m_2}^{n_2} ((n_1-1)+j) + \sum_{j=m_2}^{n_2} (n_1+j)$$

In the lecture notes,

$$\sum_{i=m_1}^{n_1} \left(\sum_{j=m_2}^{n_2} (i+j) \right)$$

is unambiguously written as

  - `sum for i = m1 to n1 of (sum for j = m2 to n2 of (i + j))`
  - `sum for i = m1 to n1 of (sum for j = m2 to n2 of i + j)`
  - `sum for i = m1 to n1 of sum for j = m2 to n2 of (i + j)`
  - `sum for i = m1 to n1 of sum for j = m2 to n2 of i + j`

- A nested sum, Version 2:

$$\sum_{i=m_1}^{n_1} (\sum_{j=m_2}^{n_2} (i+j))$$

$$\equiv \sum_{i=m_1}^{n_1} ((i+m_2)+(i+(m_2+1))+\ldots+(i+(n_2-1))+(i+n_2))$$

$$\equiv \sum_{i=m_1}^{n_1} (i+m_2+i+(m_2+1)+\ldots+i+(n_2-1)+i+n_2)$$

2 Mathematical induction

Definition 1 (Inductive characterization of natural numbers)**.** *A natural number is either* 0 *or it is the successor of a natural number.*

Definition 2 (Proof by mathematical induction)**.** *A property indexed by a natural number holds for any natural number if and only if*

base case: *it holds for* 0*, and*

induction case: *assuming that it holds for a natural number, it also holds for the successor of this natural number.*

3 Sum of the first even natural numbers

Proposition 3.

$$\forall n \in \mathbb{N},\ \sum_{i=0}^{n} 2 \times i = n \times (n+1)$$

Definition 4 (predicate notation)**.**

$$\forall n \in \mathbb{N},\ P(n) \equiv \sum_{i=0}^{n} 2 \times i = n \times (n+1)$$

Proof of Proposition 3:

Let us prove that $P(n)$ holds for any n, by induction on n.

base case: We need to show that $P(0)$ holds, i.e., that the following equality holds:

$$\sum_{i=0}^{0} 2 \times i = 0 \times (0+1)$$

The left-hand side, $\sum_{i=0}^{0} 2 \times i$, simplifies to 0.
The right-hand side, $0 \times (0+1)$, simplifies to 0.
Since the left-hand side and the right-hand side coincide, the equality holds, i.e., $P(0)$ holds. The base case is thus proved.

induction case: under the induction hypothesis $P(k)$ for some natural number k, we need to show that $P(k+1)$ holds as well.
The induction hypothesis reads:

$$\sum_{i=0}^{k} 2 \times i = k \times (k+1)$$

We need to show that the following equality holds:

$$\sum_{i=0}^{k+1} 2 \times i = (k+1) \times ((k+1)+1)$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side, i.e., $(k+1) \times (k+2)$:

$\sum_{i=0}^{k+1} 2 \times i$
= {by definition of a sum}
$(\sum_{i=0}^{k} 2 \times i) + 2 \times (k+1)$
= {using the induction hypothesis}
$(k \times (k+1)) + 2 \times (k+1)$
= {since $\times$ is commutative}
$((k+1) \times k) + (k+1) \times 2$
= {by factoring $(k+1)$ on the left}
$(k+1) \times (k+2)$

which coincides with the right-hand side.

Since the left-hand side and the right-hand side are equal, the equality holds, i.e., $P(k+1)$ holds. The induction case is thus proved.

So we have shown that $P(0)$ holds, and that for any natural number k such that $P(k)$ holds, $P(k+1)$ holds as well. Therefore $P(n)$ holds for all natural numbers n, which proves Proposition 3.

□

4 Sum of the first exponents of a natural number

Proposition 5.

$$\forall x \in \mathbb{N} \textit{ such that } x \geq 2, \forall n \in \mathbb{N}, \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

Definition 6 (predicate notation)**.**

$$\forall x \in \mathbb{N} \textit{ such that } x \geq 2, \forall n \in \mathbb{N}, \ Z(x, n) \equiv \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

Proof of Proposition 5:

Let us prove that for any given x such that $x \geq 2$, $Z(x, n)$ holds for any n. We proceed by induction on n.

base case: We need to show that $Z(x, 0)$ holds, i.e., that the following equality holds:

$$\sum_{i=0}^{0} x^i = \frac{x^{0+1} - 1}{x - 1}$$

The left-hand side reads as follows:

$\sum_{i=0}^{0} x^i$
= {by definition of a sum}
x^0
= {by definition of exponentiation of integers}
1

The right-hand side reads as follows:

$\frac{x^{0+1}-1}{x-1}$
=
$\frac{x^1-1}{x-1}$
=
$\frac{x-1}{x-1}$
=
1

Since the left-hand side and the right-hand side coincide, the equality holds, i.e., $Z(x, 0)$ holds. The base case is thus proved.

induction case: under the induction hypothesis $Z(x, k)$ for some natural number k, we need to show that $Z(x, k+1)$ holds as well.

The induction hypothesis reads:

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$$

We need to show that the following equality holds:

$$\sum_{i=0}^{k+1} x^i = \frac{x^{(k+1)+1}-1}{x-1}$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side, $\frac{x^{k+2}-1}{x-1}$:

$\sum_{i=0}^{k+1} x^i$
= {by definition of a sum}
$(\sum_{i=0}^{k} x^i) + x^{k+1}$
= {using the induction hypothesis}
$\frac{x^{k+1}-1}{x-1} + x^{k+1}$
=
$\frac{x^{k+1}-1}{x-1} + \frac{x^{k+1}\times(x-1)}{x-1}$
= {distributing x^{k+1} on the left of $x - 1$}
$\frac{x^{k+1}-1}{x-1} + \frac{x^{k+2}-x^{k+1}}{x-1}$
=
$\frac{x^{k+1}-1+x^{k+2}-x^{k+1}}{x-1}$
= {simplifying and reordering}
$\frac{x^{k+2}-1}{x-1}$

which coincides with the right-hand side.

Since the left-hand side and the right-hand side are equal, the equality holds, i.e., $Z(x, k+1)$ holds. The induction case is thus proved.

So we have shown that $Z(x, 0)$ holds, and that for any natural number k such that $Z(x, k)$ holds, $Z(x, k+1)$ holds as well. Therefore for any given x, $Z(x, n)$ holds for all natural numbers n, which proves Proposition 5.

□

Exercise 7. *Can you think of an alternative solution that does not use mathematical induction? (Hint: multiply $x - 1$ on the left and on the right of the putative equality, and simplify on the left.)*

Exercise 8. *Prove the following statement:*

$$\forall n \in \mathbb{N}, \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

5 Exponentiating a 2×2-matrix

Definition 9 (2×2-matrix multiplication). *For all numbers $x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}$,*

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \times \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} x_{11} \times y_{11} + x_{12} \times y_{21} & x_{11} \times y_{12} + x_{12} \times y_{22} \\ x_{21} \times y_{11} + x_{22} \times y_{21} & x_{21} \times y_{12} + x_{22} \times y_{22} \end{bmatrix}$$

Property 10 (2×2-matrix multiplication is associative). *For all 2×2-matrices M_1, M_2 and M_3,*

$$M_1 \times (M_2 \times M_3) = (M_1 \times M_2) \times M_3$$

Definition 11 (the identity 2×2-matrix). *The identity 2×2-matrix is defined to be* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Property 12 (I is neutral for 2×2-matrix multiplication). *The identity matrix is neutral for 2×2-matrix multiplication on the left and on the right: for any 2×2-matrix M,*

$$I \times M = M = M \times I$$

Definition 13 (exponentiation of a 2×2 matrix). *For any 2×2-matrix M,*

- $M^0 = I$
- $\forall n \in \mathbb{N},\ M^{n+1} = M^n \times M$

Proposition 14.

$$\forall n \in \mathbb{N}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Definition 15 (predicate notation).

$$\forall n \in \mathbb{N},\ E(n) \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Proof of Proposition 14:

Let us prove that $E(n)$ holds for any n, by induction on n.

base case: We need to show that $E(0)$ holds, i.e., that the following equality holds:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By definition of matrix exponentiation, the left-hand side is I.

The right-hand side is also I.

Since the left-hand side and the right-hand side are equal, the equality holds, i.e., $E(0)$ holds. The base case is thus proved.

induction case: under the induction hypothesis $E(k)$ for some natural number k, we need to show that $E(k+1)$ holds as well.

The induction hypothesis reads:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

We need to show that the following equality holds:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1}$

= {by definition of matrix exponentiation}

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

= {using the induction hypothesis}

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

= {by definition of matrix multiplication}

$\begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$

which coincides with the right-hand side.

Since the left-hand side and the right-hand side are equal, the equality holds, i.e., $E(k+1)$ holds. The induction case is thus proved.

So we have shown that $E(0)$ holds, and that for any natural number k such that $E(k)$ holds, $E(k+1)$ holds as well. Therefore $E(n)$ holds for all natural numbers n, which proves Proposition 14.

□

Proposition 16.

$$\forall n \in \mathbb{N},\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$$

Exercise 17. *Prove Proposition 16.*

Proposition 18.

$$\forall n \in \mathbb{N},\ \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}^{n+1} = \begin{bmatrix} 2^{2n+1} & 2^{2n+1} \\ 2^{2n+1} & 2^{2n+1} \end{bmatrix}$$

Exercise 19. *Prove Proposition 18.*

Proposition 20.

$$\forall n \in \mathbb{N},\ \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}^{n+1} = \begin{bmatrix} 2^{3n+2} & 2^{3n+2} \\ 2^{3n+2} & 2^{3n+2} \end{bmatrix}$$

Exercise 21. *Prove Proposition 20.*

Proposition 22.

$$\forall n \in \mathbb{N},\ \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}^{n+1} = \begin{bmatrix} 2^{4n+3} & 2^{4n+3} \\ 2^{4n+3} & 2^{4n+3} \end{bmatrix}$$

Exercise 23. *Prove Proposition 22.*

Exercise 24. *Do you see a pattern in the previous 4 exercises? Care to generalize it and to prove it?*

Exercise 25. *Let* $F \equiv \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
Calculate, by hand or using a Scheme program, the 8 first successive powers of F, *i.e.,* F^0, F^1, F^2, F^3, F^4, F^5, F^6, *and* F^7. *What do you observe? Could you prove it, and if so how?*

Exercise 26. *Let* $G \equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
Calculate, by hand or using a Scheme program, several successive powers of G. *What do you observe? Could you prove it, and if so how?*

Definition 27 (alternative exponentiation of a 2×2 matrix)**.** *For any 2×2-matrix* M,

- $M^0 = I$
- $\forall n \in \mathbb{N},\ M^{n+1} = M \times M^n$

Exercise 28. *Using this alternative definition of exponentation, re-prove Proposition 14.*

Proposition 29. *For any 2×2-matrix* M, $\forall n \in \mathbb{N}, M \times M^n = M^n \times M$, *using Definition 13.*

Definition 30 (predicate notation)**.**

$$\forall M,\ \forall n \in \mathbb{N},\ R(M, n) \equiv M \times M^n = M^n \times M$$

Proof of Proposition 29:

Let us prove that for any given M, $R(M, n)$ holds for any n. We proceed by induction on n.

base case: We need to show that $R(M, 0)$ holds, i.e., that the following equality holds:

$$M \times M^0 = M^0 \times M$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side:

$M \times M^0$
= {by definition of exponentiation}
$M \times I$
= {since I is right-neutral for matrix multiplication}
M

Let us continue from the right-hand side and reason our way towards obtaining the left-hand side:

$M^0 \times M$
= {by definition of exponentiation}
$I \times M$
= {since I is left-neutral for matrix multiplication}
M

The left-hand side and the right-hand side are equal, and therefore $R(M, 0)$ holds. The base case is thus proved.

induction case: under the induction hypothesis $R(M, k)$ for some natural number k, we need to show that $R(M, k + 1)$ holds as well.

The induction hypothesis reads:

$$M \times M^k = M^k \times M$$

We need to show that the following equality holds:

$$M \times M^{k+1} = M^{k+1} \times M$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side:

$M \times M^{k+1}$
= {by Definition 13}
$M \times (M^k \times M)$
= {since matrix multiplication is associative}
$(M \times M^k) \times M$
= {by induction hypothesis}
$M^{k+1} \times M$

which coincides with the right-hand side.

Since the left-hand side and the right-hand side are equal, the equality holds, i.e., $R(M, k + 1)$ holds. The induction case is thus proved.

So we have shown that $R(M, 0)$ holds, and that for any natural number k such that $R(M, k)$ holds, $R(M, k+1)$ holds as well. Therefore for any given M, $R(M, n)$ holds for all natural numbers n, which proves Proposition 29.

□

Exercise 31. *Prove that for any 2×2-matrix* M, $\forall n \in \mathbb{N}, M \times M^n = M^n \times M$, *using Definition 27.*

Corollary 32. *Definitions 13 and 27 are equivalent.*

Proposition 33.

$$\forall n \in \mathbb{N},\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

Exercise 34. *Using either definition of exponentiation, prove Proposition 33 by mathematical induction.*

6 Transposing a 2×2-matrix

Definition 35 (2×2-matrix transposition)**.** *For any 2×2-matrix* M,

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

its transposed version*, noted* ${}^{\top}M$, *reads as follows:*

$$\begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}$$

Property 36 (transposition is involutive)**.** *For any 2×2-matrix* M, ${}^{\top}({}^{\top}M) = M$.

Lemma 37. *For any 2×2-matrices* X *and* Y, ${}^{\top}(X \times Y) = {}^{\top}Y \times {}^{\top}X$

Proof of Lemma 37:

Let $X \equiv \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ and let $Y \equiv \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$.
Let us start from the right-hand side.

${}^{\top}Y \times {}^{\top}X$
= {by definition of matrix transposition}
$\begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{bmatrix} \times \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix}$
= {by definition of multiplication}
$\begin{bmatrix} x_{11} \times y_{11} + x_{12} \times y_{21} & x_{21} \times y_{11} + x_{22} \times y_{21} \\ x_{11} \times y_{12} + x_{12} \times y_{22} & x_{21} \times y_{12} + x_{22} \times y_{22} \end{bmatrix}$
= {by definition of matrix transposition}
${}^{\top}(X \times Y)$

□

Proposition 38. *Transposition (Definition 35) and exponentiation (Definition 13) commute with each other.*

Definition 39 (predicate notation)**.**

$$\forall M,\ \forall n \in \mathbb{N},\ Q(M, n) \equiv {}^{\top}(M^n) = \left({}^{\top}M\right)^n$$

Proof of Proposition 38:

Let us prove that for any given M, $Q(M, n)$ holds for any n. We proceed by induction on n.

base case: We need to show that $Q(M, 0)$ holds, i.e., that the following equality holds:

$${}^{\top}(M^0) = \left({}^{\top}M\right)^0$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side:

${}^{\top}(M^0)$
= {by definition of exponentiation}
${}^{\top}I$
= {by definition of transposition}
I

Let us continue from the right-hand side and reason our way towards obtaining the left-hand side:

$\left({}^{\top}M\right)^0$
= {by definition of exponentiation}
I

The left-hand side and the right-hand side are equal, and therefore $Q(M, 0)$ holds. The base case is thus proved.

induction case: under the induction hypothesis $Q(M, k)$ for some natural number k, we need to show that $Q(M, k + 1)$ holds as well.

The induction hypothesis reads:

$${}^{\top}(M^k) = \left({}^{\top}M\right)^k$$

We need to show that the following equality holds:

$${}^{\top}(M^{k+1}) = \left({}^{\top}M\right)^{k+1}$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side:

$^{\top}(M^{k+1})$
= {by definition of exponentiation}
$^{\top}(M^k \times M)$
= {using Lemma 37}
$^{\top}M \times {}^{\top}(M^k)$
= {using the induction hypothesis}
$^{\top}M \times ({}^{\top}M)^k$
= {using Proposition 29 for $^{\top}M$}
$({}^{\top}M)^k \times {}^{\top}M$
= {by definition of matrix exponentiation}
$({}^{\top}M)^{k+1}$

which coincides with the right-hand side.

Since the left-hand side and the right-hand side are equal, the equality holds, i.e., $Q(M, k+1)$ holds. The induction case is thus proved.

So we have shown that $Q(M, 0)$ holds, and that for any natural number k such that $Q(M, k)$ holds, $Q(M, k+1)$ holds as well. Therefore for any given M, $Q(M, n)$ holds for all natural numbers n and so indeed, transposition and exponentiation commute with each other, which proves Proposition 38.

□

Exercise 40. *Observing that* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ *are the transposed matrices of each other, formulate a new proof of Proposition 33 that does not use mathematical induction.*

7 Nested sums

Proposition 41. $\forall f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \forall m \in \mathbb{N}, \forall n \in \mathbb{N},\ \sum_{i=0}^{m} \sum_{j=0}^{n} f(i,j) = \sum_{j=0}^{n} \sum_{i=0}^{m} f(i,j)$

Definition 42 (predicate notation)**.**

$$\forall f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \forall m \in \mathbb{N}, \forall n \in \mathbb{N},\ C(f, m, n) \equiv \sum_{i=0}^{m} \sum_{j=0}^{n} f(i,j) = \sum_{j=0}^{n} \sum_{i=0}^{m} f(i,j)$$

Proof of Proposition 41:

Let us prove that for any given f, $C(f, m, n)$ holds for any m and n, by induction on m.

base case: We need to show that $\forall n \in \mathbb{N},\ C(f, 0, n)$ holds, i.e., that the following statement holds:

$$\forall n \in \mathbb{N},\ \sum_{i=0}^{0} \sum_{j=0}^{n} f(i,j) = \sum_{j=0}^{n} \sum_{i=0}^{0} f(i,j)$$

For any given n, the left-hand side, $\sum_{i=0}^{0}\sum_{j=0}^{n} f(i,j)$, simplifies to $\sum_{j=0}^{n} f(0,j)$, and the right-hand side, $\sum_{j=0}^{n}\sum_{i=0}^{0} f(i,j)$, also simplifies to $\sum_{j=0}^{n} f(0,j)$. Since the left-hand side and the right-hand side coincide, the equality holds, i.e., $\forall n \in \mathbb{N},\ C(f,0,n)$ holds. The base case is thus proved.

induction case: under the induction hypothesis that $\forall n \in \mathbb{N},\ C(f,k,n)$ holds for some natural number k, we need to show that $\forall n \in \mathbb{N},\ C(f,k+1,n)$ holds as well.

For any given n, the induction hypothesis reads:

$$\sum_{i=0}^{k}\sum_{j=0}^{n} f(i,j) = \sum_{j=0}^{n}\sum_{i=0}^{k} f(i,j)$$

We need to show that the following equality holds for the given n:

$$\sum_{i=0}^{k+1}\sum_{j=0}^{n} f(i,j) = \sum_{j=0}^{n}\sum_{i=0}^{k+1} f(i,j)$$

Let us start from the left-hand side and reason our way towards obtaining the right-hand side:

$\sum_{i=0}^{k+1}\sum_{j=0}^{n} f(i,j)$
= {by definition of a sum}
$(\sum_{i=0}^{k}\sum_{j=0}^{n} f(i,j)) + \sum_{j=0}^{n} f(k+1,j)$
= {by induction hypothesis}
$(\sum_{j=0}^{n}\sum_{i=0}^{k} f(i,j)) + \sum_{j=0}^{n} f(k+1,j)$
=
$\sum_{j=0}^{n}(\sum_{i=0}^{k} f(i,j) + f(k+1,j))$
=
$\sum_{j=0}^{n}\sum_{i=0}^{k+1} f(i,j)$

which coincides with the right-hand side.

Since the left-hand side and the right-hand side are equal and we made no hypothesis on the given n, the equality holds for any given n, i.e., $\forall n \in \mathbb{N},\ C(f,k+1,n)$ holds. The induction case is thus proved.

So we have shown that $\forall n \in \mathbb{N},\ C(f,0,n)$ holds, and that for any natural number k such that $\forall n \in \mathbb{N},\ C(f,k,n)$ holds, $\forall n \in \mathbb{N},\ C(f,k+1,n)$ holds as well. Therefore for any given m, $\forall n \in \mathbb{N},\ C(f,m,n)$ holds, which proves Proposition 41.

□

Definition 43. $\prod_{i=1}^{n} i \equiv 1 \times 2 \times \ldots \times (n-1) \times n$

Proposition 44. $\forall f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \forall m \in \mathbb{N}, \forall n \in \mathbb{N},\ \prod_{i=0}^{m}\prod_{j=0}^{n} f(i,j) = \prod_{j=0}^{n}\prod_{i=0}^{m} f(i,j)$

Exercise 45. *Prove Proposition 44.*

8 Nested induction

The goal of this section is to illustrate nested induction, i.e., the use of an induction proof in the course of an induction proof. To this end, here is a classical example, the Ackermann-Péter function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, which is defined as follows:

$$\begin{aligned} A(0, n) &= n + 1 && (1) \\ A(m + 1, 0) &= A(m, 1) && (2) \\ A(m + 1, n + 1) &= A(m, A(m + 1, n)) && (3) \end{aligned}$$

Proposition 46. $\forall m \in \mathbb{N}, \forall n \in \mathbb{N},\ A(m, n) > n$

Proof of Proposition 46:

Let us prove that the inequality $A(m, n) > n$ holds for any m and n. We proceed by induction on m.

outer base case: We need to show that $\forall n \in \mathbb{N},\ A(0, n) > n$ holds.

Let us start from the left-hand side, for any given n:

$A(0, n)$
= {by definition of A}
$n + 1$

Since $n + 1 > n$ for any given n, the outer base case is proved.

outer induction case: under the *outer* induction hypothesis $\forall n \in \mathbb{N},\ A(i, n) > n$ for some natural number i, or equivalently $\forall n \in \mathbb{N},\ A(i, n) \geq n+1$ for some i, we need to show that $\forall n \in \mathbb{N},\ A(i + 1, n) > n$ holds as well.

We proceed by induction on n.

inner base case: We need to show that the inequality $A(i + 1, 0) > 0$ holds.

Let us start from the left-hand side:
$A(i + 1, 0)$
= {by definition of A}
$A(i, 1)$
$>$ {by applying the outer induction hypothesis to 1}
1

Since $1 > 0$, the inner base case is proved.

inner induction case: under the *inner* induction hypothesis that the inequality $A(i+1, j) > j$ holds for some natural number j, we need to show that the inequality $A(i + 1, j + 1) > j + 1$ holds as well.

Let us start from the left-hand side:
$A(i + 1, j + 1)$
= {by definition of A}
$A(i, A(i + 1, j))$
$\geq$ {by applying the equivalent outer induction hypothesis to $A(i + 1, j)$}
$A(i + 1, j) + 1$
$>$ {by applying the inner induction hypothesis to j}
$j + 1$

The inequality holds, and the inner induction case is thus proved.

So we have shown that the inequality $A(i+1,0) > 0$ holds, and that for any natural number j such that the inequality $A(i+1,j) > j$ holds, the inequality $A(i+1,j+1) > j+1$ holds as well. Therefore the inequality $A(i+1,n) > n$ holds for all natural numbers n. The outer induction case is thus proved.

So we have shown that $\forall n \in \mathbb{N},\ A(0,n) > n$ holds, and that for any natural number i such that $\forall n \in \mathbb{N},\ A(i,n) > n$ holds, $\forall n \in \mathbb{N},\ A(i+1,n) > n$ holds as well. Therefore the inequality $\forall n \in \mathbb{N},\ A(m,n) > n$ holds for all natural numbers m, which proves Proposition 46.

□

The young gulls looked at him quizzically.
Hey, man, they thought, this doesn't sound like a rule for a loop.
Fletcher sighed and started over.
"Hm. Ah... very well," he said, and eyed them critically.
"Let's begin with Level Flight."

– Richard Bach[1]

[1] *This is some geeky movie reference, isn't it?*[2]

– Jorge Cham

[2] This one isn't. It is from a book. CONSTANT VIGILANCE!

– Alastor "Mad Eye" Moody